The Relationship Between LC and Canonical Foundations of Mathematics

Dimensional Systems
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Since the LC model incorporates an account of mathematical concepts, and since this account differs in certain respects from the canonical one, it may be useful here to explore the historical and theoretical background of these ideas. In his *Tractatus Logico-Philosophicus*, Wittgenstein writes, "Mathematics is a logical method,"[1] and here, as in other respects, the LC model agrees with much of his conception. The goal is not a reduction of one of these terms, math or logic, to the other (as with logicism and some species of set theory), but instead a delineation of their cognate modes of reasoning. The emphasis is always on method: in this case, on the stepwise formation of the number concepts and their concomitant calculi.

The LC approach to number theory directly evolves out of our account of type orderings: viz., that the instances of certain types (nominal ones) are relatively unordered with respect to one another, while others (ordinal and cardinal ones) are characterized by explicit orderings. To be sure, nominal instances can be ordered any way we like, e.g., alphabetically or by load sequence. The point is that these orderings are extraneous to the meanings of the instances. Ordinal and cardinal relations, on the other hand, are internal to, even constitutive of, the meanings of their respective instances. When such is the case, the LC model prefers recourse to constructive methods and notation for expressing the series of instances.

A constructive methodology must be carefully disentangled from a constructive metaphysics, the latter being readily identified with the intuitionists, and with the idea that numbers and algorithms are properly characterized as the counterparts of certain mental constructions. Constructive *methods*, however, have their origin within classical mathematics – in the proofs of *both* Kronecker and Cantor – and in fact most schools have employed them. Within the context of the model, "constructive" simply means the use of recursively defined procedures or algorithms, whose application results in a serially constructed sequence.

Of course, the successful application of this method to the number concepts would carry with it a metaphysical lesson: one about the dispensable nature of extra-mathematical foundations (at least so far as these have been treated by the various schools grouped around the foundations issue). Since Cantor's introduction of his *mengenlehre,* and Frege's reduction of the concept "number" to that of "class", the old question of "what is number" has been successfully deferred by appeal to this more ambiguous category. Further inquiry into the nature of sets or classes usually divides along three interpretive stances (corresponding to three metaphysical dispositions), according to whether 1) the foundational paradigm requires our commitment to the existence of these sets (Platonism); or 2) as it reduces the concept of class to some more acceptable ontology, e.g., individuals or tokens (formalism); or 3) as it rejects any mind-independent constituents of the mathematical universe (intuitionism).[2]

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[1] Wittgenstein (1972), proposition 6.2. (Hereafter cited in the text as *TLP*.)

[2] See Thomsen/Shavel (1993), 208. Cf. Quine (1961), 14 where he relates the three schools to their medieval counterparts (realism, nominalism and conceptualism, respectively), according to their degree of ontological commitment.

The LC approach we have elsewhere characterized as a "functional" account of symbolic representation: that what makes a sign into a symbol is how it is used in a particular application.[3] This approach displaces debate over the *existence* of numbers by examining the role of number-concepts within the calculi, working outward to delimit the mathematically determinable. As a general heuristic it is kindred to Wittgenstein's methodology.

Returning to the type attributes now, it will be noticed that one feature distinguishing the ordinal and cardinal cases from nominative ones is the way instances of the first two types may be ordered by recursive iterations of an operation. Repeated applications of an operation yield a series of successive instances whose order is that succession.[4] It represents a simple ranking: first, second, third, etc. What distinguishes a cardinal series from an ordinal, then, is the further stipulation that the iterations be of a fixed base, or metric. A constructive series needs to specify this metric unit before we can pose questions of 'how much?' or 'how many?'. Only then can we assert "in 2nd place, by a length", or "15 days ago". Here horse-length and solar day are our measure constants presupposed in such statements.

Less obvious, in fact almost unrecognized, is how metric plays a role, not only in measurement (i.e. in an application of mathematics), but also in the specification of the pure number concepts. Metric unit is implicit to a number series in the form of a unit magnitude (so that 2 is twice the base unit, etc.). When we abstract or construct a purely ordinal series we may safely ignore the concept of unit: the 2nd and 3rd iterations are not related by multiples of a common magnitude, only by a concept of sequence. But in a cardinal series, such as the naturals, magnitude is a defining character of the sequence. It is what allows us to apply the series directly, in a measuring or counting context. (So within the number concepts, magnitude is not to be identified with a measure; or simply put, measure is the application of a magnitude.)[5]

This entire take on the number concepts is in sharp contrast to the standard account. The various theories we have called canonical can be characterized in three general ways: 1*) They ignore metric*. The differences between number concepts are treated according to pure cardinality differences between infinite and transfinite classes. This is seen as a positive feature, the result of Cantor and Dedekind's stated program to fully arithmetize the continuum. 2) *They generalize from the empirical case*. Numbers are defined as classes of classes of things in the world: hence the need for an axiom of infinity postulating an infinity of things. (So, on this account, mathematical propositions are lent a higher degree of certainty by virtue of their "scope," i.e. their degree of generality.) 3) *They generalize from the finite to the infinite case*. This is premised on the set-theoretic methodology, which treats of collections finite and infinite alike as completed domains. It traces back to Cantor's univocal notation for all classes, and what

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[3] Thomsen/Shavel (1990, 1993).

[4] Whatever operations one employs (e.g., Skolem's recursive arithmetic), the notion of series or sequence is not meant to imply some appeal to temporal intuition; it is intended to make perspicuous the internal relation linking the operands. To this degree it is immaterial whether one pictures the iterations as serial (i.e., successive), or massively parallel (i.e., simultaneous). See Wittgenstein's closing remarks in *Remarks on the Foundations of Mathematics,* V, § 51. (Hereafter cited in the text as *RFM*.)

[5] The concept of metric proposed here is not entirely without historical antecedents. Cf. Aristotle's arguments for the primary relationship of "measure" to "unit" and "quantity" (*Metaphysics*, X, (1052[a], 18-24)).

Michael Hallett calls his Principle of Finitism: "the transfinite is to be treated as far as possible like the finite." (Hallett, 7)

Wittgenstein writes

> "Generality in mathematics is a direction, an arrow pointing along the series generated by an operation…. The infinite number series is itself such a series – as emerges from the single symbol [1, *x*, *x* + 1]. This symbol is itself an arrow with the first '1' as the tail of the arrow and '*x* + 1' as its tip." (Wittgenstein 1964, § 142) (Hereafter cited as *PR*.)

The symbol he gives here, in the *Philosophical Remarks*, has its provenance in the *Tractatus*, and bears closer examination. *TLP* 5.501 offers three general methods for specifying a collection, or stipulating the values of a variable:

> 1. Direct enumeration, in which case we can simply substitute for the variable the constants that are its values; 2. Giving a function *fx* whose values for all values of *x* are the propositions to be described. 3. Giving a formal law that governs the construction of the propositions, in which case the bracketed expression has as its members all the terms of a series of forms.[6]

The first two of these are of course the traditional specifications of a class by extension (listing its members) and intention (giving a property common to all those members). Because mathematics deals with infinite collections, where enumerating members isn't possible, it is the second method that has come to characterize the canonical account of number. Since Frege, numbers are defined by generalizing on the properties of sets: so the number two, e.g., is the most general and peculiar property of all the sets of pairs. The third method is particular to the *Tractatus*. Here the variable is equated with the rule for constructing its values. The distinction between the latter two expressions has its roots in the important (though neglected) tractarian distinction of function and operation, which is worth examining.

The relevant features of this distinction are roughly as follows:

—that operations, like functions, are given by variables (5.24);[7]

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[6] It will be noticed we are using the terms 'set' and 'class' interchangeably. 'Set' is usually given as a restricted version of 'class'; but the reasons for these restrictions are precisely the ones addressed in the LC model. Also, we will accept Russell's "no class" translation of class-membership symbols to propositional functions, so that we may refer to predicate logic and set theory interchangeably (i.e., such that any F*x* can be expressed as $x \in F$ and vice versa.

[7] The tractarian characterization of these terms is intended as a refinement of their customary usage in mathematical logic, which in turn borrows from their mathematical meaning. E.g., an operation is "a particular method of defining a function of a given set of independent variables" (Post, 10). A function is equivalent to the dependent variable in mathematics. Its "dependence" is why a function , unlike an operation, is not recursive. Wittgenstein writes, "The reason why a function cannot be its own argument is that the sign for a function [e.g., F*x*] already contains the prototype of its argument, and it cannot contain itself" (*TLP*, 3.333). "Prototype" (*Urbild*) is synonymous with primitive logical form (3.315). In *TLP*, logical forms correspond to use-distinctions in a language or calculus - not types of things in the world. "A sign determines a logical form only when account is taken of the ways in which it may combine with other signs according to the rules of logical syntax" (Black, paraphrasing *TLP* 3.327). In the LC model there are two primitive prototype distinctions: *Type / instance* and *location / content*. In the latter case, e.g., we can only assert a content of a location (never a content of a content): this is what it means to be a content (to specify the prototype of its argument).

—that, like functions, they do not characterize the content ('sense') of their operands (5.25);
—that, unlike functions, they characterize the difference between forms, not the forms themselves (5.241);
—that unlike functions, they are recursively iterable (and so, self-applicable) (5.251, 5.242);
—that they determine a "formal series", that is, any series governed by an internal relation (4.1252), (as opposed to functions, which are given by an external relation or property);
—that "formal concepts" (e.g., the concepts 'class' or 'number,' as opposed to the concept 'lion') can only be "shown", and are expressed by a variable (4.126); and
—that the bracketed notation for an operation, such as the one generating a number-series, is such a variable, and so indicates the prototype for a formal concept (5.2522).

The general term of a formal series is given at 5.2522 as [*a*, *x*, O'*x*]. In this notation *a* symbolizes the unit metric, which is the first term of a series; *x* is a variable standing in for any term of the series, while the operation O'*x* indicates how succeeding terms are generated. Number is finally defined as "the exponent of an operation" (*TLP* 6.021). So, for example, the (positionally) inductive symbol quoted earlier (1, *x*, *x* + 1) yields the series '1 + 1 + 1…', such that any natural number is equivalent to the finite number of iterations over the base metric that yields it. But the range of the operation itself is non-terminating.

(This expression, given in the *Philosophical Remarks*, is amended from the one formulated at *TLP* 6.03 – [0, ξ , ξ + 1] – where Wittgenstein was likely trying to capture the Peano axioms. The later notation is preferred because it emphasizes the concept 'unit' or 'metric' as a starting point, over the concept 'term without predecessor'. This accords more closely with the LC approach. The place of zero in the system would then be defined by the addition of a simple rule, say 'x – x = 0'. The number systems have always been home to such apparently ad hoc rules – e.g., that division by zero is meaningless – rules that by their very familiarity have lost their normative character for us. Wittgenstein stresses this normative role when he insists on their "arbitrary" relation to one another.[8] But what is not arbitrary is the relation between rules and the calculi they determine.)

The two criteria, then, for an operation constructing a cardinal series are first, that it should indicate a metric, and second, that it should make prespicuous an internal relation holding between the operands – and this it does by simply showing how from any instance we can generate its successor. So in addition to the usual definitions by extension and intention, Wittgenstein specifies what we may call a definition by "progression," applicable to numbers and all non-terminating series whose members are internally related. It's primitive nature is shown by the fact that a progressional definition cannot be reduced to an intentional definition without forfeiting certain defining characteristics – namely those concerning the connectivity

[8] See Wittgenstein, (*RFM*) I, § 167; (1967), § 352; (1974), 184 (hereafter cited as *PG*). Also Shanker's illuminating discussion, 315 ff.

among its members. Or to put it another way, the suppression of these characteristics in the classical notation is one of the most serious stumbling blocks facing the canonical account.[9]

The unfortunate impression left by *TLP* 5.501 is that all three methods for specifying a collection are on equal logical footing – a notion Wittgenstein explicitly rejected later on. According to Moore's record of the 1932 lectures, "he said that, when he wrote the *Tractatus*, he had supposed that *all* such general propositions were 'truth-functions'; but he said now that in supposing this he was committing a fallacy, which is common in mathematics, e.g. the fallacy of supposing that 1 + 1 + 1 + … is a sum, whereas it is only a *limit*, and that dx / dy is a quotient, whereas it is also a *limit*" (Moore 298). It is not all clear that he really did hold this view in the *Tractatus*. In any case, the LC logical model rejects quantification over bracketed expressions. We treat them as expressing a rule, rather than a truth-function. As a result we still maintain the extensional equivalency of the predicate and propositional calculi (that is, the identity of the quantified (*x*) F*x* with the logical product Fa and Fb and Fc, etc.; and (*Ex*)F*x* with the logical sum Fa or Fb or Fc, etc.)

In empirical cases, the range of *x* may be unbounded and possibly infinite – but that would be an empirical matter to decide. In other cases, such as with the construction of number concepts, the infinity is something stipulated – within the language – not something discovered. It belongs to a logico-syntactical form, given by the law governing its construction. And it is symbolized by expressing the rule, not by postulating a completed domain subject to quantification. Consequently Wittgenstein concludes, "The theory of classes is completely superfluous in mathematics. This is connected with the fact that the generality required in mathematics is not *accidental* generality." (*TLP* 6.031).[10]

Later he is more specific:

> It seems to me that we can't use generality – all, etc. – in mathematics at all. There's no such thing as 'all numbers', simply because there are infinitely many. And because it isn't a question here of the amorphous 'all', such as occurs in 'All apples are ripe', where the set is given by an external description: it's a question of a collection of structures, which must be given precisely as such (PR § 126).[11]

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[9] The attempt to express this orderly togetherness of numbers as part of the definition of an inductive number led to Frege and Russell's application of the "ancestral" relation, (for a discussion of which, see Appendix II). It was essentially an attempt to capture the inheritance of number properties (Peano's fifth axiom) in the classical notation of functions and quantifiers. For Wittgenstein, application of this notation to the definition of a (properly speaking) formal concept was doomed to failure: it "contained a vicious circle" (*TLP* 4.1273).

[10] Cf. 6.1232, "The general validity of logic might be called essential, in contrast with the accidental general validity of such propositions as 'All men are mortal'."

[11] Cf. *PR* 129, "I have always said you can't speak of all numbers, because there is no such thing as 'all numbers'." This is not a finitist assertion about how many numbers exist, but a restatement of the tractarian argument against the extensional and intentional representations of a formal concept (4.1272). Formal concepts (such as 'number', 'class', 'proposition') indicate logico-syntactical types – forms of linguistic usage, or operands of a calculus. As such they are not names for an extension and carry no cardinality. (For more on formal concepts, see Appendix II and Thomsen/Shavel (1993), 208).

Intentional definitions are indifferent as to whether a descriptive property is accidental or necessary, external or internal to its extension, so long as it is uniquely specifying. This indifference is married to a further ambiguity, as to whether the intention ranges over finite or infinite collections. This can have its own advantage for most applications. In many cases we want to be able to form expressions such as ‘All apples are ripe’ without having to know in advance just how many apples there are, or whether the set of apples is finite or infinite. It can, however, be a disadvantage when we are treating of stipulated infinites, such as the number systems. This is because the non-terminating extendibility of, say, the natural numbers is not something fortuitous, something we happen upon in the course of counting off the instances (whether or not – historically or pedagogically – this is how it dawns on us, in the case where we can't seem ever to arrive at an end to our counting). Rather this infinitude is part and parcel of what it means to be a (natural) number.

The argument is that a formally inductive expression of the number concepts is *conceptually* prior to the set theoretic account by description. This should not be taken as relying on or implying an *historical* priority. It would be irrelevant to the argument whether human societies first encountered the number two inductively, say through the practice of counting one's flock; or by a process of abstraction, by discovering, say, “that a brace of pheasants and a couple of days were both instances of the number” (Russell 1920, 3). At issue is only our claim that ‘2’ takes its meaning from its place in the system of natural numbers, that that system circumscribes a calculus. Which means that our knowledge at 2's position between 1 and 3 also involves a knowledge of how one many move between this place and any other, compare their relative magnitudes, etc. And these features are more basic to our understanding of the number than any notion (however valid) that two is the distinguishing property common to the set of all pairs.

Further, it is necessary to know in the (logical) type structure that the number of numbers is non-terminating. This is not represented by an infinite extension of elements, nor by external stipulation (as, say, with an axiom of infinity), but again, by giving the rule whereby we can see how elements may be generated without limit.[12] The requirement for an axiom of infinity[13] in the canonical account of number is based on the need to assert that the domain of numbers really is infinite. This must be postulated as an additional condition, since it is not deducible from the symbolic expressions themselves (set membership or quantified functions). But more

[12] For this reason so-called “primitive” number systems that contain finite elements are not correctly or even meaningfully comparable with these progressions. Our naturals do not “extend” or “amend” the mathematics of a culture that, e.g., only counts to five, or from five to “many”. Such a finite counting system is adequate to itself, and most likely to the needs of that culture. Compared to it our mathematics is simply different, not “better”. (Historically, we may be permitted to talk of an extension, even of an evolution. But as in the biosphere, a new species is discrete and autonomous: and the evolution of “and so on” from “many” is as distinguishing as the first appearance of a species to live on land, or take to the air.

[13] “If *n* be any inductive cardinal number, there is at least one class of individuals having *n* terms… Since *n* is any inductive number, it follows that the number of individuals in the world must (if our axiom be true) exceed any inductive number.” (Russell, 1920, 131)

significantly, it derives from the extra-mathematical underpinnings of the canonical frameworks: because numbers are defined by generalizing over individuals, we require an inexhaustible supply of individuals in the world from which to form the classes of mathematics. Russell is forthright on this point: “Suppose there were exactly nine individuals in the world …Then the inductive cardinals from 0 up to 9 would be such as we expect, but 10 (defined as 9 + 1) would be the null-class …The same will be true of 9 + 2, or generally of 9 + *n*, unless *n* is zero. Thus 10 and all subsequent cardinals will all be identical, since they will all be the null-class” (Russell, 1920, 132). And this would be in non-compliance with the third Peano axiom, viz., that no two numbers have the same successor.

Of course, the infinity of individuals is not at all certain, and certainly not a priori knowledge. It is, if determinable at all, something to be decided by physics or metaphysics. But such a finding is irrelevant, “accidental,” to the infinity of number concepts,[14] which is a priori: that is to say, this infinity is determined within the calculus by a rule – or moreso, the rule is constitutive of the calculus itself. It is these considerations that led Wittgenstein to the conviction that the infinity of the number concepts belongs to the “grammar” or logical syntax of the calculus. So in discussion with the Vienna circle, he states, “A correct symbolism has to reproduce an infinite class in a completely different way from a finite one. Finiteness and infinity of a class must be obvious from its syntax” (Waismann 227).

This is precisely what the canonical apparatus of classes, predicates and quantifiers is incapable of capturing. The arbitrary and external nature of class specification is the main target for Wittgenstein’s critique of the quantifier and of existential proofs in mathematics (they amount to a willed ignorance of the algorithm relating terms (*RFM* V, passim).

Syntactically speaking, the LC model specifies two independent directions, or modes of traversal, along which infinite series may be generated, corresponding to the two primitives of adjacency-linking: position and resolution. We can think of a positional traversal as a function of changes in the multiple of some base unit. Resolutional traversal involves quantitative changes in the base unit of some multiple, as in the decimal expansion .1111.... Specification of the first sort alone captures the natural numbers, as shown above. Incorporating the second captures the rationals.[15]

Although positional and resolutional series are orthogonally related, they always need to be co-specified. That is, specification in either series is relative to a fixed value of the other series. Co-specified links are always embedded in ones that are unspecified, and meaning is built out in tandem, from given metric units. On the other hand, co-specification of number adjacencies is, to borrow Russell’s phrase, “systematically ambiguous”, in that we don’t need to define, for example, the smallest resolutional units to “build up” the scale. And just as,

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[14] See TLP 5.535. Or are we to believe that the certainty of mathematical propositions is to be put on hold until the confirmation of this uncertain axiom?

[15] We cannot define the rationals like the naturals, by a serial progression, since there is no first rational successor of a rational. This isn’t a drawback - it’s the reason a resolutional infinity is orthogonal to a positional one. Between any two rationals, however close, an inexhaustible number of divisions is possible. But we can express them as ratios of natural numbers (m/n) or by decimal expansions.

functionally speaking, neither positional nor resolutional iteration presupposes a completed domain for its meaning (the actually-infinitely large or the actually-infinitely small), so neither mode of iterating requires the completed domain of the other to operate along. Changing position relative to a fixed resolution, and changing resolution relative to a fixed position are mutually orthogonal. No extension of the naturals will ever define any fractional successor, and no extension of the fractions or decimals will ever define any natural successor.

To say that resolutional adjacencies are orthogonal to positional ones, means that an operation along either axis is functionally independent of the other. Only loosely speaking, then, does a positional traversal of the number line from 0 to 1 "pass through" the infinity of resolutional subdivisions along the way. It is by a conflation of these modes of traversal (such as occurs in a number of Zeno's and Bolzano's paradoxes of the infinite) that one gets the misleading impression of our having to complete an infinite series with each positional iteration.[16]

But what about the irrationals? The Pythagorean discovery that the diagonal of a square with sides of length 1 does not determine a rational magnitude on the number line led, by the circuitous route of history, to Dedekind's observation that "the line *L* is infinitely richer in point-individuals than is the domain *R* of rational numbers in number-individuals."[17] Next Cantor's transfinite number theory was envisaged as a univocal methodology for differentiating the real and rational numbers according to differences of cardinality between their domains. At last, it seemed, the number concepts could be treated in a fully arithmetized context. But how valid is this arithmetized picture of missing numbers on the rational line[18]? And how are we guided by this picture in comparing the sizes of infinite domains?

One support for this approach comes from the ideal of global operations in mathematics. The goal of achieving closure under the operations of arithmetic (+, –, ×, ÷, etc.) leads us to the idea that ∨*x* must yield a number for any numerical value of *x*, else the calculus would be open. It would not be "formalizeable", that is, describable in a formal symbolism without regard to the meanings or "semantics" of the symbols. This latter notion is part and parcel of the referential framework within which mathematical logicians and philosophers of mathematics have come to interpret their notation. Let us take an informal look at this formal requirement.

Partly by appeal to a reification – that of having devised a concrete symbol for the roots (∨) – and partly by analogy to a certain reciprocity among the basic operations, an inadequacy in the rational number system is diagnosed. For just as we are compelled to see a recursive symmetry between $x + y = z$ and $z - y = x$, and $x \times y = z$ and $z \div y = x$ for any value, it seems incumbent upon us to recognize as well an operational reciprocity between $x^2 = y$ (surely global

[16] The preceding two paragraphs are amended slightly from Thomsen/Shavel (1993), p. 213.

[17] Trans. And cited in Dauben, p. 48.

[18] There are two extreme forms of metric definable in a 2-Type 'XY'schema composed of hierarchical cardinal types (fractional integers or 'Rationals'). They are mutually exclusive; Viz. Delta X XOR Delta Y which is the basis for Euclidian metrics (e.g. movement at any level of granularity is along edges), and Delta X IFF Delta Y which is the basis for Polar metrics. Thus the irrationality of for example the square root of 2 has nothing do to with any kind quantity supposedly missing from the rationals but rather the definitional inability within a Euclidian metric to move simultaneously in X and in Y which is what defines the diagonal.

in its application) and ∨$y = x$. But how global, how formally reciprocal, is this algebra? We accept, almost as a given, that $0 \times y = 0$; but $y \div 0$ is considered meaningless: and this is something that must be *stipulated*, not inferred. Why then does one feel "compelled" to believe that $x^2 = 2$ must be solved for a number (any more than, say, $1^x = 2$)? We say that something multiplied by itself = 2. And to whatever our degree of sophistication at calculating, we approximate a value for x to so many decimal places. But the picture that x has an absolute or fixed value "at infinity" may be just that: a picture, and one with extra-mathematical foundations.

In the standard account, the signs for basic arithmetic are given a referential interpretation: they refer to discrete operations, which are then applied – externally – to the various types of number. These number-types more or less adequately allow the full potential of the operations to be expressed. So, for example, the natural numbers do not allow a larger number to be subtracted from a smaller, and this was viewed as a shortcoming The integers emend this situation by "extending" the naturals into the negative numbers. Now the '–' of subtraction is said to be global in its scope; for any integer values, $x - y = z$, and arithmetic is said to be closed under subtraction.

On a functional interpretation, however, the symbol '–' only has its meaning in the context of a system – i.e. a particular calculus; and it is misleading to interpret it as referring to some autonomous and given meaning. This is because, constructively speaking, numbers and operations are determined at the same time, by the same construction. The construction *is* the system, or to put it the other way around, a system is always "traceable back to an operation" (Wittgenstein, quoted in Waismann 1979, 216). The bracketed expression giving the naturals, e.g., shows how the numbers are co-specified with addition.[19]

The key to the distinction of mathematical and logical propositions on the one hand, and empirical propositions on the other, lies in the fact that both meaning and truth for the first sort are specified within a calculus. Wittgenstein writes, "the system of rules determining a calculus thereby determines the 'meaning' of its signs too. Put more strictly: The form and the rules of syntax are equivalent. So if I change the rules – seemingly supplement them, say – then I change the form, the meaning" (PR §152). The ideal of global operations may be guiding as an heuristic device, but it should not guide us in the mistaken belief that non-global operations are deficient, or that the calculi in which they operate are inadequate. Every operation in mathematics is exactly adequate to its calculus – precisely because the two are co-determined. Within one autonomous system $n - (n + m)$ has meaning, within another it doesn't. In other words, not only the values for n and m differ between the integers and the naturals, but the "meaning" of 'minus' as well.

Just as with closure for subtraction, closure under the roots can only be claimed when we have constructed an arithmetical rule, or finite set of rules,[20] for generating or extracting them.

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[19] The other operations on the naturals can be constructively devolved from '+,' as has been demonstrated in a number of ways. (See, e.g., Waismann (1951), p. 27ff). As with the logical constants (see Appendix II), interdefinability shows that these are not names for discrete, extra-systemic relations.

[20] Wittgenstein draws an important distinction between the demand that the series of numbers must be surveyable (in other words, finite), and the demand that the laws by which they are generated must be. His is not the finitist's concern that our operations must be completable i.e., capable in theory or practice of being carried out. (See

Perhaps a closer analogy can be drawn to the imaginary numbers, whose "extension" of the number concept was also motivated by closure under the roots. (Here, for solving cubic equations, and simple algebraics like $x^2 + 1 = 0$: i is postulated as the solution for x, that is, the square root of negative one). Again, an inadequacy was perceived that needed shoring up because in the integers as given, there are no roots for negative numbers: because it is a rule that a negative number multiplied by a negative number yields a positive. (We might ask, what would the integers look like if this were not the rule - if, say, the product were negative? And what would become of the imaginaries?)

It is not within the scope of this paper to question the merits of postulating new types of number every time a lack is perceived. What is significant in the case of the imaginaries, however, is that no attempt is made to compare them in terms of magnitude to the integers. Mathematicians are assiduous on this point. In Gauss' geometrical treatment they are represented on a *y*-axis orthogonal to the number line. Complex equations are shown as combining numbers from the two axes, much as one plots an *x*, *y* coordinate on the Cartesian plane: but the orthogonal independence of the axes prevents us from "squashing" these equations onto the *x* axis: that is, the Gaussian treatment doesn't allow us to determine a complex magnitude on the number line. "For example, which of the two numbers $2 + 3i$ and $3 + 2i$ is the greater? The (linear) order is not valid and therefore neither is the concept of 'betweenness'" (Waismann, 1951, 11).

Another hallmark of a constructive approach is that it allows us to show in what manner and to what degree the autonomous number systems are related to each other. In the LC model, differences between number systems are explained by differences in the mode and manner of iterating over a metric. The naturals are expressed as a uni-directional iteration over a base unit metric; the integers as a bi-directional iteration.[21] Both are examples of a positionally iterative series. To this degree, we can see how much "overlap" exists between these autonomous systems. (That is, how much sense there is in comparing 2 and +2, e.g.). The imaginaries represent a different type of integer, corresponding to different rules of calculation: to this extent we can see how only nonsense would result from comparing these integer types. Next the decimals are expressed as quantitative transforms on the base unit itself. They define a mode of resolutional iteration. An orthogonal co-specification of the positional and resolutional series, then, gives us a purely logico-syntactical representation of the rationals.

The irrationals, however, present us with a unique challenge. Wittgenstein writes, 'The confusion in the concept of the 'actual infinite' arises from the unclear concept of irrational number, that is, from the fact that logically very different things are called 'irrational numbers' without any clear limits being given to the concept (*PG* 471).[22] This is because on an LC

Shanker, 128ff.) We might put it this way: mathematical infinity *is* the case where operations are non-terminating; and this is something that must show itself, in the notation for that operation.

[21] In the naturals, subtraction follows from addition. You can only take as much as has already been (inductively, or incrementally,) specified. In the integers, incrementation and decrementation are equally primitive directions of operating.

[22] We have been arguing here, as we have elsewhere, that there is greater continuity between Wittgenstein's early and later thought than is commonly recognized. And this seems especially true for of his philosophy of mathematics. In one respect however, that continuity is not at all obvious. The later work (especially *Remarks on the Foundations of Mathematics*) stresses the autonomy of the various number systems, while the *Tractatus* would seem to imply (if only by its focus) that arithmetic is basic, and that all higher mathematics devolve from the natural

account, the various mathematical notions grouped under the number-concept “irrational” devolve from heterogeneous and incommensurate metrics. The concept of incommensurate metric turns on the idea that there are both quantitative and non-quantitative aspects of metric. For example the decadic divisions of a meter stick (into centimeters, millimeters, etc.) involve proportional quantitative transformations on a metric unit. Quantitative transforms yield a resolutional scale, such as account for the rational subdivisions of the number line. Hence the co-specification of positional and resolutional adjacencies provides a fully “syntactical” representation of the number line: given an arbitrary unit abscissa from the origin we can generate all whole number and rational intervals. Our purpose now is to show that the interference of “missing” (i.e., irrational) intervals involves a confusion of the syntactical with the “semantic” (or geometrical) number line.

Fundamentally, there are two extreme forms of metric definable in a 2-Type ‘XY’schema composed of hierarchical cardinal types (fractional integers or ‘Rationals’). They are mutually exclusive; Viz. Delta X XOR Delta Y which is the basis for Euclidian metrics (e.g. movement at any level of granularity is along edges), and Delta X IFF Delta Y which is the basis for Polar metrics. Thus the irrationality of for example the square root of 2 has nothing do to with any kind of quantity supposedly missing from the Rationals, but rather the definitional inability within a Euclidian metric to move simultaneously in X and in Y which is what defines the diagonal.

Differing metrics yield differing representations of measure. Cartesian space, e.g., represents the infinite extension in both a positional and a resolutional direction of two or more dimensions of straight (i.e., Delta X XOR Delta Y) intervals such that the angle between any two dimensions is 90 degrees. Polar systems result from the combination of both angle- and distance-based measuring dimensions (i.e., Delta X IFF Delta Y, viz. a series of lines intersecting at a common point, that serves as the common center to a series of circles of differing radii).

Angle-changes and distance-changes can both be arbitrarily small. This is why π can never be exactly specified in a Cartesian metric. For every increase in precision in a Cartesian metric-based measuring tool, there is the measurement-based realization that a circle will continue to exhibit a curved behavior – because it is cut from a different metric. (Circling the square is as impossible as squaring the circle. It is not just the endpoint of a circumference that lacks definition on a line.)

The approach we have outlined suggests an alternate program for determining completability of the real numbers, according to arbitrarily-differentiable metrics, each according to arbitrarily-differentiable positional and resolutional adjacencies. On this approach, difficulties associated with the continuum problem, classically defined, begin with its first presupposition: namely, Cantor’s and Dedekind’s ‘axiom of continuity’, which asserts that for every real number there corresponds a definite point of the line whose (distance-based) magnitude is equal to that number. Recall the functional distinction drawn earlier within the concept of linearity: whether the term ‘line’ is being used syntactically (as the token for representing a something), or semantically (as the something representable by tokens). If ‘line’ is being used as a token, it

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numbers and their concomitant operations. To address this question fully would require an entire paper in its own right. Here we can only assert, based on our own interpretation: the iterative construction of the integers in *TLP* is the primary focus because it *shows* what is basic to all mathematical calculi: viz., the presence of a formal concept (metric), and of a procedure for iterating over that concept. Differences in either the metrics or in operating, then, are what account for the variety of number concepts (and their corresponding calculi, proof structure, etc.)

only needs to have a one-one or cardinal relationship with the real number continuum. If 'line' is being used as a thing, its relationship with the continuum needs to be one-one and continuous.

Metric provides the interface between these conceptions. Philosophers of mathematics have tended to conflate these distinctions, treating syntactical as identical with semantical linearity: and semantically the line is a pure extension of a 180 degree, or distance-based, metric. No metric can exactly specify the iterations of an other, orthogonal metric. The line as an extension of a straight unit path is unable to differentiate the orthogonal iterations of a rotational unit path.

The LC approach takes exception with the notion of real number, to the extent that this concept treats the numbers of autonomous and orthgonally constructed systems (natural, rational and irrational) as belonging to a single over-arching system, and to the extent that they are represented in a univocal manner. This characterization of the problem is not an attempt to convince a working mathematician that she/he should toss $\sqrt{2}$ on the trash heap. But we are calling into question the idea that its status on the real number line is compulsory, unquestionable, or self-evident (such that Cantor and Dedekind can *initiate* their investigation of the reals with an axiom that assumes this idea).

Descartes' exact specification of a geometric circle in terms of an algebraic equation has been interpreted as showing, or at least helping to show, that algebra has a descriptive power equal to and independent of geometry. Foundational studies have subsequently pursued the closure of the number-concepts in terms of pure, arithmetic relations; given the correlations of analytical geometry, geometrical concepts are assumed to follow *tout court*. (What has been overlooked in this account is that a coordinate system is not "transparent" to what it describes, and that the algebraic specification of a circle is itself a function of the geometrical form of the metric used to measure the circle – Cartesian or polar, e.g.) The next critical step came with Cantor's reduction of multi-dimensional to linear point sets, which he viewed as necessary for his and Dedekind's program to "arithmetize" the continuum, and so provide a solely quantitative basis for distinguishing the denumerable (everywhere dense) rationals, from the more inclusive, non-denumerable points on the real number line.

The irony is that this very impulse – the desire to remove geometry from the scene – tacitly relies on a geometric picture for its support: the comparison of real number magnitudes pictured as points and/or cuts along a one-dimensional line. And it follows from an equally geometric way of fitting those points or cuts there. As Wittgenstein observes, "It is *by combining calculation and construction*" – i.e., by the rotation of the diagonal of the unit square onto the number axis – "that one gets the idea that there must be a point left out on the straight line" (*RFM* IV, § 37).[23] It is not by leaving the domain of the rationals that one encounters the real points: it is by leaving the domain of arithmetic.

It should be clear from the foregoing that, while a one-one mapping from planes to lines may be cardinally successful, such a translation will involve the suppression of crucial (geometric) properties: in the case of the square, collapsing the diagonal suppresses angle differentiation, which defines the relation of the diagonal to the sides. Likewise, it seems to

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[23] It seems as if we can translate this idealized measurement to an applied measure situation, where we might for instance talk about using the same ruler for measuring the sides and the diagonal of some "exact" but actual unit square. However, in this less-than-ideal case we *can* given an exact length. To whatever the degree of accuracy allowed by our measuring device (e.g., to the nearest millimeter, micron, whatever), we will of course specify an accurate rational distance: somewhere along the expansion of 1.41421....

follow intuitively that "unrolling" the circumference of a circle with diameter 1 should yield a point on the real number line. According to the foregoing, however, π cannot be so specified; in fact, it involves the extreme case of reducing to a linear, or distance-based interval specification, a magnitude belonging to another (purely rotational) metric. Our attempts to do so will merely generate translation rules; i.e., rules for constructing non-terminating resolutional sequences on the number line. This resolutional expansion is simply the result of conflating magnitudes derived from incommensurate metrics.

Our purpose here is not to reduce arithmetic again to a species of geometric construction, but to demonstrate the geometric assumptions implicit to the purely quantitative treatment of the real numbers.

The set-theoretic account of geometry's relation to higher mathematics again leans on the notion of generality. In this characterization, the development of set theory is portrayed as the end result of successive generalizations on the figure-bound "metric" geometries. The evolution from metric to affine to projective geometries brings us closer to the most abstract features (e.g., distortion-invariance) of topology. Finally, dimensional mappings onto the continuum leave us with the most general, most purely arithmetic of relationships: the cardinality differences between finite, infinite and transfinite point sets. (See, e.g., Waismann, 1951, 177ff.) The point sets themselves are related according to degrees of generality, with the "missing" irrationals at last accounted for in the all-inclusive domain of the transfinite sets.

On an LC account, mathematics is more adequately characterized by its distinguishing features of logical syntax, than by the generality of its scope. The functional objection to set theory, here as elsewhere, is that a methodology based on arbitrary intentional specification is being employed to group together logically disparate sorts of things. In this case the picture of an over-arching concept ("real number") is used to run together the rationals (that is, the iterative and quantitative transforms on a base metric) with the specifications of non-proportional metrics. What gets lost in the shuffle is that "irrational numbers" have no place among rationals independent of a construction. Here we mean construction in the broadest sense, regardless of whether we are speaking in geometric terms (as in the squashing of curved and linear magnitudes) or in algebraic ones (as in the construction of convergent series of fractional or decimal expansions). Functionally speaking, construction is distinguished from number as operation from iteration. If we have characterized this difference correctly, then at the same time we have provided a general heuristic for replacing the notion of irrational *numbers* with the notion of rule-governed procedures for generating (non-terminating) sequences of rational intervals.

These logico-syntactical distinctions are precisely the ones that canonical notation is incapable of capturing. This is because the concept-scripts are inherently nominal in character: the relation of set to members, or predicate-function to arguments, is oblivious to any relationships between members or argument-values. The all-embracing descriptive power of the mathematical logic is premised on the generalizing character of its notation: first, the totalizing character of its variables, treated as general names for an extension, and second, the arbitrariness of giving an extension by an external description.

Perhaps the most disturbing feature of the Cantor / Dedekind program is the way it turns a vagueness into a virtue. Generally speaking it is the ambiguous relation of a variable to its extension that is enshrined in the Principle of Finitism (Hallet, *ibid*), by which a symbolism devised for representing finite totalities may be co-opted for expressing infinite series. More specifically, the ability to define a class by any uniquely specifying intention gives rise to a very curious distinction of infinite from finite sets. An infinite set is "defined" as any set which may be put into one-one correspondence with a proper subset of itself. For any finite series of integers, (say, 1-100) there will be fewer even numbers, (by half) than total numbers in the set. Not so in the infinite case. When we correlate all the naturals by $m = 2n$, we find there are just as many evens as evens and odds put together.

The genesis of this definition has an interesting history. Similar one-one correlations had been performed on the infinite before – but usually as part of a *reductio*, to show the absurdity that comes of comparing the cardinalities of infinite sets. The practice traces back to "Galileo's paradox", whose original interpretation has been glossed over since set theory adopted one-one correspondence as its principal method for counting and comparing infinities. As Shanker reminds us:

> Galileo had actually set out to show that 'the attributes "larger," "smaller," and "equal," have no place either in comparing infinite quantities with each other or in comparing infinite with finite quantities.' To demonstrate this, he effected a one-one correspondence between the natural numbers and their squares in order to show that 'There are as many squares as there are numbers because they are just as numerous as their roots, and all the numbers are roots.' Yet this conferred an inexplicable puzzle: the latter series certainly appears to be 'smaller' than the former – 'at the outset we said there are many more numbers than squares, since the larger portion of them are not squares' – so how could the interstitial sequence of squares possibly be as 'large' as the sequence of the natural numbers? His conclusion, however – contra subsequent developments – was that 'the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities.' (Shanker 177)

In set theory the paradoxical result is no longer cautionary: because any property, even an inexplicable one, can be used to specify an intention. It is simply stipulated as the definition of infinite class. Of course if there really is something paradoxical involved in this application, we might expect that it be at odds with some grounding principle of mathematics. And indeed, in the conflict between this definition and the axiom – extant since Euclid – that a whole is greater than a part of itself, it is the latter that is abandoned. Deprived of this axiom in the context of the infinite we can no longer employ the relations '>', '<': only '='. Parts are "equal" to wholes. But what does this '=' mean divorced from the comparisons of magnitude, the calculus of >, < and = that characterizes all number systems? Procedurally, the sign means "correlated one to one" – which by an analogy to finite correlations, is taken to mean "equinumerous". But is the analogy adequate? In the finite case '=' is functionally defined, on a scale of magnitude; it is part of the mathematical "meaning" of the sign that to say two classes are equinumerous, or that one number is equal to another, is also to be able to say how many that is.[24] Moreover, functionally speaking, '=' only has meaning in the context of a calculus, and we can't presume that its significance in one calculus will carry over into another, entirely foreign context.

The situation is further complicated in the case of the transfinite number classes: so much so that we can only touch on its most general features here. Among the transfinite classes the invalidated relations '>', '<' are again re-introduced, as necessary to the well-ordering of the continuum. (That is, we need to say $\aleph_1 > \aleph_0$ if we want to say that the infinite reals are "larger"

[24] For a fuller discussion of these ideas, see Shanker 178ff.

in size than the infinite rationals, and so account for the missing numbers on the rational line.) On any finite class we can employ a power-set operation to construct the more numerous set of all its subsets: so for a set of *n* members, we can always generate 2^n subsets. Transfinite number theory turns on the conceit that the same operation applied to an infinite class yields the same ratio of set to powerset. In other words, while $2\aleph_0 = \aleph_0$ (as in the denumerable correlations discussed before) $2^{\aleph_0} > \aleph_0$.[25]

The proofs that the power set of an infinite set has a higher cardinality, or that the set of reals is larger than the set of naturals, are markedly different than the quasi-constructive correlations that Cantor uses to show that certain infinite sets are denumerable (i.e., equinumerous with the natural numbers). In fact it is a curious property of all proofs of denumerability that they are similarly constructive, while non-denumerability proofs favor indirect methods and *reductio ad absurdam*. With this in mind, and by way of conclusion, we will glance at Cantor's 1891 "diagonal" procedure for showing that the set of reals is non-denumerable: because it is the "textbook" proof, the one most often used to establish transfinite inequalities, and because it sheds a clarifying light on the nature of indirect proof.

To set up the proof, Cantor first assumes the completed domains of two number-sets: the naturals, and the reals between a certain interval; second postulates, *ex hypothesi*, a function uniquely correlating members of the sets; and third, constructs a real number within the interval not included in the correlation. The "missing" number is constructed as follows: the reals are represented by an array of non-terminating decimals (randomly given, since the expansions are infinite and cannot be ordered). Because the array includes rational expansions (some terminating, some non-terminating, repeating), finite decimals are succeeded by an infinite series of zeroes. The naturals are given ordinally, opposite each decimal. Next, a new number, *d*, is constructed such that it intersects with one digit of each decimal expansion (the first of the first, the second of the second, etc.), and such that it is different from each (e.g., every digit of *d* is specified to be '1', unless the digit it intersects is 1, in which case it becomes '2'). The new number then differs from the *k*th decimal at the *k*th digit, and from all of them at ∞. One is led to conclude that one-one correspondence fails, and that the sets are not cardinally equatable (and further, that the reals, containing at least one uncorrelatable member, belong to a higher power.) Wittgenstein writes:

> 'These considerations may lead us to say that $2^{\aleph_0} > \aleph_0$.' That is to say: we can *make* the considerations lead us to that. Or: we can say *this* and give *this* as our reason. But if we do say it what are we to do next? In what practice is this proposition *anchored*? It is for the time being a piece of mathematical architecture which hangs in the air, and looks as if it were an architrave, but not supported by anything and supporting nothing. (*RFM*, Appendix II, § 8).

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[25] We are focusing here on the transfinite cardinals and ignoring, for reasons of conciseness, the situation among the transfinite ordinals, in which the '>' governs not only the relation between ν, the series of natural numbers (a series which has no greatest number), and ω, the lowest ordinal after the naturals – but also governs all the various relations between transfinite ordinals, some ordered by the first, and some by the second principle of generation. Shanker writes, "It is difficult, to say the least, to keep track of all the relations that are operating here, all under the same guise of '>'." (Shanker 173)

The objection Wittgenstein raises here is that the transfinite calculus, whatever the merits of its construction or proof structure, remains unconnected to the edifice of mathematics as a whole. It is not functionally related to the other number concepts – except perhaps by an (extra-systemic) appeal to the ambivalent ‘>’.[26] These qualms may not be convincing to those who see in transfinite set theory a workable way of accounting for the irrationals, within a cardinally ordered taxonomy of finite, infinite and transfinite classes.[27] But how does the proof support this account, and where does the “extra” real number *d* fit into this taxonomy? Clearly, *d* is not accounted for by the difference between the finite and infinite decimals, since the rationals (even before adding all those zeroes) include infinite expansions (i.e., non-terminating, repeating decimals like .333...): And the rationals, as is well known, were shown by Cantor to be denumerable (i.e., correlatable to the naturals).

But then neither is it accounted for by the difference between the rationals and the irrationals. This is because the algebraic numbers, as is less well known, were also shown to be denumerable – also by Cantor himself, using quasi-constructive correlations. And the algebraic numbers (the numerical roots of polynomial equations with rational coefficients) include a great many of the irrationals: √2, for example. This latter finding by itself should call into question the Cantor / Dedekind program for treating the difference between rational and irrational according to the cardinality differences between infinite and transfinite classes. Apparently, the denumerably infinite class $\aleph_0$ – the “number” of the naturals – is powerful enough to account for quite a few of the irrationals (an infinity, in fact): leaving it for a subset of the irrational numbers, the “transcendentals”, to account for “gaps” in the number line, and to fill out the first transfinite number class.

Wittgenstein’s criticism of the Canton proof (for which he has himself been criticized) centers not on the procedure of diagonalization, but on the cardinality lesson drawn from it. What the proof shows – that presuming the fixed extension of two infinite domains we can, in Poincaré’s words, “disrupt the correspondence” (68) between them – contravenes what it says it shows (i.e., that the reals are more numerous than the naturals).

It is possible to vindicate Wittgenstein’s and Poincaré’s critique in the following manner, by constructing an alternative – “looking-glass” – diagonal proof with different results. We begin with Cantor’s schema, which as we have seen gives the natural numbers in vertical order down the page. First we prefix every finite natural quantity with a non-terminating precession of zeroes, turning it into an infinite numeral. But, and this is important, we are not turning it into an infinite number: as with our succession of zeroes in the decimal array, this precession does not alter the value of a number. Nor does it alter the number of naturals on our list (as is obvious: the horizontal expansion of zeroes does not affect the vertical extension of the column). We do this merely to ensure that our new diagonally defined number always encounters a numerical digit.

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[26] Cf. *PG* 464: “An infinite class is not a class which contains more members than a finite one, in the ordinary sense of the word ‘more’. If we say that an infinite number is greater than a finite one, that doesn’t make the two comparable, because in that statement the word ‘greater’ *hasn’t the same meaning* as it has say in the proposition 5 > 4!”

[27] Cantor has written, “One can say unconditionally: the transfinite numbers stand or fall with the finite irrational numbers: they are alike in their innermost nature, since both kinds are definitely delimited forms or modifications of the actual infinite.” (Trans. and cited in Hallet, 80).

Next we randomize the order in which the naturals are disposed in the array. This is sanctioned by Cantor’s own definition of cardinality (as abstracted from all properties of a set, including the ordinal), and likewise his thesis that the cardinality of any set (finite, infinite, transfinite) is independent of its ordering (Cantor 86; Dauben 157). We do this to ensure that our new diagonally defined number has an equal chance of encountering any random digit at every point of intersection in its expansion. Assuming, then, *ex hypothesi* a one-one bi-jection of the sets, we can construct a “missing” natural, *d´*, thereby “proving” the set of naturals is larger than the set of reals:

Naturals	*Decimals <0,1>*
$\ldots b_{1v} \ldots b_{13}, b_{12}, b_{11}$	$.a_{11}, a_{12}, a_{13} \ldots a_{1v} \ldots$
$\ldots b_{2v} \ldots b_{23}, b_{22}, b_{21}$	$.a_{21}, a_{22}, a_{23} \ldots a_{2v} \ldots$
.	.
.	.
.	.
$\ldots b_{vv} \ldots b_{v3}, b_{v2}, b_{v1}$	$.a_{v1}, a_{v2}, a_{v3} \ldots a_{vv} \ldots$
.	.
.	.
.	.

$$d' = \sim [\ \ldots b_{vv} \ldots b_{22}, b_{11}]$$ [28]

The first objection one might raise to this so-called proof is: how do I know that the left side of this array still represents the naturals? Well, this was stipulated, *ex hypothesi*, just as was with the right side. But we can further strengthen the claim by enumerating the bi-jections themselves, top to bottom. A second objection might be, how can *d´* be a natural number if it is infinitely long? But in setting up the proof we showed how all the finite numbers can be expressed as infinite numerals. On its own this is not a sufficient objection; but it is perhaps closer to the mark. One must further object, how do I know all infinite numerals in this construction represent naturals?[29] Of course, in the case of all the infinite numerals except *d´*, we do know, as already shown. And we could respond to this by asking, if *d´* isn't a natural number, what sort of number is it? When did it extend further than the other infinite numerals, and leave the natural domain? But the point of the construction isn't really to prove there are more natural numbers than the given set of reals. It is a *reductio* on a *reductio*, and it only works to the degree that one accepts the symmetry of construction between this and Cantor's proof. (What we would really like you to ask is, how do I know that Cantor's *d* represents a real number? – and not simply halt before the counter-question, if it isn't a real, what sort of number is it?)

In other words, the difficulties with the diagonal number are symptomatic of both sides of its construction. These are 1) *d* and *d´* are empirical constructions. They are determined according to whatever numerals they happen to encounter, and that depends on how we happen to have randomly listed the array. It is akin to constructing a number by the flipping of a coin:

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[28] This "proof" and the following discussion of it are slightly amended from the version given in Thomsen/Shavel (1993) 217f.

[29] In other words, having shown that all the finite numbers can be represented as infinite numerals, we have to show that these are *all* the infinite numerals, otherwise *d´* might belong to some set larger than the subset of infinite numerals representing finite numbers. That is, we would have to prove that there are no more infinite numerals than there are natural numbers. If we really wanted to pursue this legerdemain further, we could show that all infinite numerals can be made to correspond to a progression. First, we postulate all infinite numerals in an array that looks like the left side of our counter-Cantor proof. Next, we arrange all the numerals in this array in a series: for instance, starting with b_{11} we "serpentine" along the path b_{12}, b_{21}, b_{31}, b_{22}, b_{13} . . . . It is clear that no numeral is left out of the progression. And since all progressions are denumerable (see Russell, 1920, 83), every infinite numeral can be correlated with a natural number, and therefore so can *d´*. (Of course, you could also diagonalize on this array, proving the opposite. But that's precisely the point: it all depends on which sleight-of-hand game you want to play with infinite sets.)

yes, this construction is law-like, but the law is hardly a part of mathematics. (Is every *numeral* we can construct by whatever means we choose, a *number*?) 2) They are infinite numbers in a curious sort of way. The diagonal number – on both sides of its calculation – is only different from the numbers it intersects. Any finite calculation of its expansion has a numerical equivalent further down on the list – and therefore *isn't d*, in a much stronger sense than say π, (which we can call, for e.g. 'determinate to the *n*th place' without contradictory consequences). 3) They require "all" numbers of an infinite domain be listed before their construction. It is this conceit – of treating a non-terminating series as a totality, as a completed domain – that allows us to disrupt the array with a new number. But since we have already assented, ex hypothesi, that the "all" was all, the new number must be more than all, or more than any hypothesized correlation with that domain.

The point of this exercise is simply to show that Cantor's *reductio ad absurdum* may indicate a different absurdity than the one proposed: for instance, the summation of a non-terminating series, or alternately, a method of calculating some number that, by definition, isn't in any way determinate until the ∞th place. The difficulty with indirect proof is not, as the intuitionists maintain, that the principle of non-contradiction is inapplicable to the proof-structure of mathematics. Rather it is that the contradictory terms of hypothesis are often underdetermined, or obfuscated by the language surrounding a proof.

> If it were said: "Consideration of the diagonal procedure shows you that the concept 'real number' has much less analogy with the concept 'cardinal number' than we, being misled by certain analogies, are inclined to believe", that would have a good and honest sense. But just the opposite happens: one pretends to compare the 'set' of real numbers in magnitude with that of cardinal numbers. The difference in kind between the two conceptions is represented, by a skew form of expression, as difference of extension. (*RFM*, Appendix II, § 3).

Earlier we argued that differences between number-concepts are more adequately understood as differences of type, than of extension. Their representatives, formal concepts, are symbolized by variables whose domains prescribe a metric – while notions of cardinality are subsumed under the iterative operations determining both positional (integer-like) and resolutional (rational-like) adjacencies. The formal concepts themselves, however, carry no cardinality, and any attempt to so regard them (such as Cantor's "canonical representatives") results in representations that disrupt their own mathematical multiplicity (*TLP* 4.141, 4.1272). This follows as much from inferring 'the set of all natural numbers' to be its own cardinal number, as it does from the paradox-generating 'set of all sets' (a set than which there can be none greater, which, by application of the power-set operation, yields an allegedly yet-greater cardinality).

In the LC model, infinity is treated as a non-terminating series rather than a completed domain. And a simple construction shows that two infinite series cannot be one-one compared – because their mutual inexhaustibility forestalls any such comparison.[30] In a functionally specified account of the mathematical concepts, collection and process are as easily

[30] Here's a quasi-constructive recipe for testing whether or not the reals are denumerable: Take one bucket with all the real numbers. Take another bucket with just the naturals. Pull two numbers, one from each bucket. Repeat this procedure until one or the other bucket runs out.

distinguished as variable and operation. In the canonical account, everything is collection: and this is sanctioned by the referential reading of a variable's domain.[31] It is important to note that the position we have outlined here is not strictly finitist: the concept of infinity, insofar as this is mathematically representable, we indicate completely by the non-terminating recursion inherent in the symbol for an operation. It is significant, however, that on our account, all iterations of a metric are rational, and that irrational magnitudes are derived from the conflations of incommensurate metrics.

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[31] In Appendix II we will more fully contrast the functional vs. the referential approaches. But in the present context it is worth quoting Cantor's justification for the actual infinite in mathematics:

> There is no doubt that we cannot do without variable quantities in the sense of the potential infinite; and from this can be demonstrated the necessity of the actual infinite. In order for there to be a variable quantity in some mathematical study, the 'domain' of its variability must strictly speaking be known beforehand through a definition. However, this domain cannot itself be variable, since otherwise each fixed support for the study would collapse. Thus, this 'domain' is a definite, actually infinite set of values. Thus each potential infinite, if it is rigorously applicable mathematically, presupposes an actual infinite. (Cantor, trans. and cited in Hallett, 25)

References and Further Reading

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